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Note

Rational Approximations for Zeros of Bessel Functions

The zeros of Bessel functions play an important part in computational physics, e.g., in Fourier–Bessel series expansions [1] and in the numerical evaluation of Hankel transforms [2]. Asymptotic expansions of large zeros (e.g., McMahon's expansion [3]) or of zeros of high order Bessel functions (e.g., Olver's expansion [3]) are available, but in practical computations we need mainly small and moderate zeros of low order Bessel functions.

In [3, 4] extensive tables of the zeros of Bessel functions are presented. Although tables are very interesting, approximations are more suited for use in automatic computing.

In [5], approximations of the zeros of Bessel functions $J_a(x)$, $Y_a(x)$, $J'_a(x)$, $Y'_a(x)$ (for non-negative real a) are presented and a higher order Newton iteration formula for the accurate computation of the zeros is described.

The purpose of this note is to present rational minimax approximations of all positive zeros of $J_n(x)$, $n = 0, 1,..., 10$, which are, for most purposes, sufficiently accurate.

We denote the sth positive zero of $J_n(x)$ by $j_{n,s}$ and we consider the approximation

$$j_{n,s} \simeq \beta + \frac{a_{0,n} + a_{1,n}\beta^2 + a_{2,n}\beta^4 + a_{3,n}\beta^6}{\beta + b_{1,n}\beta^3 + b_{2,n}\beta^5 + b_{3,n}\beta^7}, \tag{1}$$

where $\beta = (s + n/2 - \frac{1}{4})\pi$ and $s = 1, 2,..., \infty$.

Using the algorithm for discrete minimax approximation given in [6], the coefficients $a_{i,n}$ and $b_{i,n}$ in (1) can be computed so that

$$\max_{1 \leqslant s < \infty} \left| j_{n,s} - \beta - \frac{a_{0,n} + a_{1,n}\beta^2 + a_{2,n}\beta^4 + a_{3,n}\beta^6}{\beta + b_{1,n}\beta^3 + b_{2,n}\beta^5 + b_{3,n}\beta^7} \right| = \text{minimal}. \tag{2}$$

The practical use of the minimax approximation algorithm requires in (2) a finite range for s, $1 \leqslant s \leqslant N$, instead of $1 \leqslant s \leqslant \infty$. Here N can be chosen so that the approximation (1) is minimax not only on $\{s | s \text{ integer}, 1 \leqslant s \leqslant N\}$ but also on the infinite set $\{s | s \text{ integer}, s \geqslant 1\}$. The value of N must be determined experimentally. A good choice is $N = 100$.

The values of $a_{i,n}$ and $b_{i,n}$ and of the maximal absolute error (left hand side of (2)) are tabulated in Table I, for $n = 0, 1, 2,..., 10$.

The maximal error of the approximation (1) increases with n, for $n \geqslant 2$. It is equal to 0.53×10^{-13} for $n = 0$ and equal to 0.96×10^{-9} for $n = 10$. For most applications

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TABLE I

Coefficients of the Rational Approximation for the Zeros of $J_n(x)$ and Maximal Error of the Approximation

n	k	a_k	b_k	Maximal error
0	0	0.68289 48973 49453 E − 01	0.10000 00000 00000 E + 01	0.53 E − 13
	1	0.13142 08074 70708 E + 00	0.11683 72425 70470 E + 01	
	2	0.24598 82418 03681 E − 01	0.20099 11221 97811 E + 00	
	3	0.81300 57215 43268 E − 03	0.65040 45772 61471 E − 02	
1	0	−0.36280 44057 37084 E + 00	0.10000 00000 00000 E + 01	0.14 E − 13
	1	0.12034 12790 38597 E + 00	−0.32564 17908 01361 E + 00	
	2	0.43945 45471 01171 E − 01	−0.11745 34459 68927 E + 00	
	3	0.15934 00884 74713 E − 02	−0.42490 69026 01794 E − 02	
2	0	−0.22782 51401 89018 E + 00	0.10000 00000 00000 E + 01	0.12 E − 13
	1	0.14517 51807 24627 E + 01	−0.10835 63523 31768 E + 01	
	2	−0.31685 26608 86333 E + 00	0.14866 93675 85701 E + 00	
	3	−0.22576 35358 37823 E − 01	0.12040 72191 13563 E − 01	
3	0	−0.13422 51923 09714 E + 01	0.10000 00000 00000 E + 01	0.18 E − 13
	1	0.11267 99400 83026 E + 01	−0.40662 54708 03507 E + 00	
	2	−0.16795 78525 61050 E + 00	0.45324 35839 14444 E − 01	
	3	0.65888 61258 99828 E − 02	−0.15060 25430 62937 E − 02	
4	0	−0.19530 79600 60853 E + 01	0.10000 00000 00000 E + 01	0.11 E − 12
	1	0.13823 69423 71281 E + 01	−0.29915 88348 31988 E + 00	
	2	−0.14062 47233 35796 E + 00	0.21722 41835 42020 E − 01	
	3	0.35038 06762 73155 E − 02	−0.44492 78428 85591 E − 03	
5	0	−0.32392 43061 63611 E + 01	0.10000 00000 00000 E + 01	0.11 E − 11
	1	0.12798 96014 23895 E + 01	−0.17088 66638 47913 E + 00	
	2	−0.74613 27315 80330 E − 01	0.72524 38647 73186 E − 02	
	3	0.10859 66295 82017 E − 02	−0.87754 85218 58614 E − 04	
6	0	−0.48749 03621 80022 E + 01	0.10000 00000 00000 E + 01	0.84 E − 11
	1	0.12197 32078 62001 E + 01	−0.11079 51536 66952 E + 00	
	2	−0.46312 03094 41842 E − 01	0.30955 79481 70757 E − 02	
	3	0.44322 38766 99637 E − 03	−0.24795 74135 12789 E − 04	
7	0	−0.68352 45458 60212 E + 01	0.10000 00000 00000 E + 01	0.40 E − 10
	1	0.11814 27883 13251 E + 01	−0.77788 53462 50360 E − 01	
	2	−0.31559 22791 01836 E − 01	0.15395 56390 58778 E − 02	
	3	0.21360 03530 57895 E − 03	−0.87630 91404 13571 E − 05	
8	0	−0.91403 04162 25556 E + 01	0.10000 00000 00000 E + 01	0.14 E − 09
	1	0.11519 36343 27892 E + 01	−0.57457 94030 43209 E − 01	
	2	−0.22756 85600 31866 E − 01	0.84559 80980 18967 E − 03	
	3	0.11438 73164 53198 E − 03	−0.35886 21688 74119 E − 05	
9	0	−0.11806 42868 43394 E + 02	0.10000 00000 00000 E + 01	0.39 E − 09
	1	0.11270 49353 33175 E + 01	−0.44019 84985 40247 E − 01	
	2	−0.17076 31457 68638 E − 01	0.49923 78534 25049 E − 03	
	3	0.66097 43222 99617 E − 04	−0.16370 88100 58130 E − 05	
10	0	−0.14845 85436 62853 E + 02	0.10000 00000 00000 E + 01	0.96 E − 09
	1	0.11052 25523 93046 E + 01	−0.34690 86081 00654 E − 01	
	2	−0.13212 05355 43942 E − 01	0.31175 99261 15802 E − 03	
	3	0.40510 95288 57727 E − 04	−0.81224 96775 99789 E − 06	

this accuracy is sufficient. If higher accuracy is required, this approximation can be improved using an iterative process for determining zeros of functions.

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